MAPPING HYPERSETS INTO NUMBERS

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Introduction

Sets are a ubiquitous data structure which gets employed, in Computer Science as well as in Mathematics, since the most fundamental level. A key feature of standard sets is the acyclicity of the membership relation upon which they are based. All in all, this is the feature enabling one to provide an inductive description of the *universe* of all (hereditarily finite) sets.

Dropping acyclicity can be desiderable, as it discloses the possibility of a direct exploitation of sets in entirely new modelling approaches (e.g., to process-algebras, to logics of actions, to networks of web pages, etc.), but it can have delicate consequences. For example, the extensionality principle ("two sets are equal if and only if they have the same elements") becomes... circular. Nevertheless, an axiomatic treatment of circular sets is possible, even though at the price of making the description and visit of the full universe of circular sets relatively cumbersome.

One rationale for the easy manageability of the universe of standard hereditarily finite sets lies in the existence of an isomorphism between it and the collection of all natural numbers, discovered by Wilhelm Ackermann around 1937 (see [Ack37, Lev79]). In this paper we consider the Ackermann bijection $\mathbb{N}_A$, the Ackermann order $\prec$, and the strong link they create between sets and numbers [TG87, Sections 7.5, 7.6]. $\mathbb{N}_A$ is a noticeable bijection between the family HF of all hereditarily finite sets and $\mathbb{N}$, which is best understood through its characteristic property:

(1) *the binary representation of* $\mathbb{N}_A(a)$ *has a 1 in position* $\mathbb{N}_A(b)$ *if and only if* $b \in a$.

As for $\prec$, it is a strict 'anti-lexicographic' ordering of all hereditarily finite sets which extends, and is isomorphic to, the standard ordering of $\mathbb{N}$.

The hereditarily finite sets to which the above encoding and order are devoted live, though, in von Neumann's celebrated *cumulative hierarchy*, in which one is allowed to define functions and relations by recursion on the membership relation; e.g. (here F and F' are restrained to be hereditarily finite sets):

$$(2)\qquad \begin{array}{rcl} \mathbb{N}_A(F) & =_{\mathrm{Def}} & \sum_{h\in F} 2^{\mathbb{N}_A(h)}, \\ F \prec F' & \Leftrightarrow_{\mathrm{Def}} & \max_{\prec}(F \setminus F') \prec \max_{\prec}(F' \setminus F). \end{array}$$

Our aim in the ongoing is to provide a similar identification of hereditarily finite sets with numbers when the universe of sets is conceived differently; namely, in agreement with the proposal of Aczel who, around 1985, investigated the consequences of superseding regularity by an *anti-foundation axiom*: *AFA* [Acz88].[1]

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[1][BM96, p. 5] indicates [FH83] as a precursor of Aczel's set theory.

Stated simply, *AFA* guarantees the existence of a unique *set-decoration* of any sensible graph, where a set-decoration is a labelling of the nodes of a graph by sets through which membership corresponds to the edge relation. The *AFA*-universe extends the standard one, which in its turn simply consists of all labels that enter in the decorations of acyclic, well-founded graphs. While disrupting the hierarchical structure of von Neumann's universe of sets by enriching it with a host of new entities (at times called '*hypersets*' [BM91]), AFA also avoids overcrowding the universe, by enforcing bisimilarity as a criterion for equality between sets. Its greater richness eases the modeling of circular phenomena, with special success when bisimilarity is at work. Typical situations of this nature are associated with automata, Kripke structures, communicating systems (cf. [DPP04]).

But many familiar definitions need to be reformulated for Aczel's sets. In particular one possible reformulation of HF, under AFA, consists in defining the family $\overline{\mathsf{HF}}$ by

$$\overline{\mathsf{HF}}(\,F\,) \quad \Leftrightarrow_{\mathrm{Def}} \quad \mathsf{Is\_finite}(\ \mathsf{trCl}(\,F\,)\ ),$$

with:

$$\mathsf{trCl}(X) \quad =_{\mathrm{Def}} \quad \textstyle\bigcup_{n\in\mathbb{N}} X\Vdash_n, \text{ where } X\Vdash_0 \,=_{\mathrm{Def}} X \quad \text{and} \quad X\Vdash_{i+1} \,=_{\mathrm{Def}} \bigcup(X\Vdash_i).$$

The set $\overline{\mathsf{HF}}$ comprises, in addition to what belongs to HF, an infinitude of hypersets. In this paper, we will see that $\overline{\mathsf{HF}}$ has a natural counterpart in the set of the so-called *dyadic rational numbers* (see below), very much like HF has its natural counterpart in $\mathbb{N}$. Specifically, we will extend Ackermann's bijection $\mathbb{N}_A : \mathsf{HF} \longrightarrow \mathbb{N}$ to $\overline{\mathsf{HF}}$, trying to preserve the strong link it creates between sets and numbers. The wording of $\mathbb{N}_A$ given in (1) directly suggests its main properties. For example, to see the bijectivity of $\mathbb{N}_A$ the following ingredients suffice:

- a simple reading—from right to left—of the digits of n, in order to inductively determine the *extension* of the set a such that $n = \mathbb{N}_A(a)$;
- to define a simple recursive routine manipulating sets and integers in binary representations to build $n = \mathbb{N}_A(a)$ from any given hereditarily finite set a.

Starting with the Ackermann function $\mathbb{N}_A$, we try to extend it to an encoding $\mathbb{Q}_A$ from $\overline{\mathsf{HF}}$ to a larger codomain $\mathbb{Y} \supset \mathbb{N}$ in such a way as to maintain the above properties of $\mathbb{N}_A$ along with property (1). We shall assign *integer positions* to hereditarily finite sets by a bijection $\mathbb{Z}_A : \overline{\mathsf{HF}} \longrightarrow \mathbb{Z}$ defined via an extension of the Ackermann order $\prec$ to $\overline{\mathsf{HF}}$. Then, using $\mathbb{Z}_A$, we will obtain a *(dyadic) rational number* $\mathbb{Q}_A(a)$ as code of a and we shall rely on the following analogue of (1):

(3) *the binary representation of $\mathbb{Q}_A(a)$ has a 1 in position $\mathbb{Z}_A(b)$ if and only if $b \in a$,*

with both $\mathbb{Z}_A$ and $\mathbb{Q}_A$ extending $\mathbb{N}_A$.

1. Extending the Ackermann order to $\overline{\mathsf{HF}}$

As argued in [LS99], an extension of the Ackermann order to $\overline{\mathsf{HF}}$ cannot be carried out naively on the basis of the antilexicographic criterion (2). For, if a and b satisfy $a = \{b\}$ and $b = \{a, \emptyset\}$ (the existence of such sets a, b ensues trivially from *AFA*), then (2) would imply that $a \prec b$ holds if and only if $b \prec a$, a contradiction.

A less naive attempt starts with the idea of extending an inductive characterization of the Ackermann order $\prec$ of HF, grounding it on the *splitting technique* devised in [PT87], which was subsequently refined—to cite two among many—in [DPP04] and [PP04]. The splitting technique is an ingredient of an algorithm for computing bisimilarity on a graph. We can set it to work in constructing the Ackermann order on HF as follows. Initially, all we know is that sets can be compared referring to the usual notion of rank for well-founded (hereditarily finite) sets:

$$\mathsf{rk}(a) < \mathsf{rk}(b) \to a \prec b.$$

At this stage, elements of the same rank are indiscernible. After this first decision has been taken, new comparison criteria naturally arise for our sets. E.g., we know that the empty set $\emptyset$ is smaller than any other set, that $a = \{\emptyset\}$ is the next set in the order, while $b = \{\emptyset, a\}$ and $c = \{a\}$ are not comparable yet. However, $b \setminus c$ owns an element which is comparable and bigger than any element of $c \setminus b$, and hence we may refine our judgement by declaring that $c \prec b$. We can then proceed in the same way, ordering the first non-trivial class of indiscernible elements at each step, and at the limit we will obtain the Ackermann order.

In the present work, the splitting technique is used to impose an order on $\overline{\mathsf{HF}}$ (this idea has been already exploited in [LS99], where the resulting order fails, though, to be an extension of the Ackermann order). In analogy with the well-founded case, we need a notion of rank for $\overline{\mathsf{HF}}$-sets, so that rank-comparison can be used as a first approximation of the sought ordering. As we are after an extension of the Ackermann order on HF, this new rank must extend the usual rank function, defined on well-founded sets. Moreover, as sets can now be *circular*, greater care is needed to ensure that the splitting process comes to an end.

Define an x, y-PATH of length $\ell \geqslant 0$ to be a sequence $x_0 = x, x_1, \ldots, x_\ell = y$ such that

1. $x_i \neq x_j$, for all $i, j \in \{0, \ldots, \ell\}$, $i \neq j$;
2. $x_{i+1} \in x_i$, for all $i \in \{0, \ldots, \ell - 1\}$.

Then, the DEPTH $\overline{d}(x, y)$ of y relative to x is the maximum length of an x, y-path, if any.

Definition 1. Let $x \in \overline{\mathsf{HF}}$. The RANK $\mathsf{rk}(x)$ of x is

$$\max(\{\overline{d}(x, y) : y \in \mathsf{trCl}(\{x\})\}).$$

Lemma 1.1. *There exists a computable function $r(n)$ which gives an upper bound for the number of elements of $\overline{\mathsf{HF}}$ having rank smaller than n.*

Using the generalized notion of rank and the splitting technique described above, it is possible to define an Ackermann order on $\overline{\mathsf{HF}}$ as follows. We first build a sequence $(\mathcal{Y}^n)_{n \in \mathbb{N}}$ of ordered partitions $\mathcal{Y}^n = \{Y_i^n : i \in \mathbb{N}\}$ of $\overline{\mathsf{HF}}$, where each partition $\mathcal{Y}^{n+1}$ is an ordered refinement of $\mathcal{Y}^n$. The $\mathcal{Y}^n$'s are constructed inductively, starting with $\mathcal{Y}^0 = \{Y_i^0 : i \in \mathbb{N}\}$ where $Y_i^0 = \{x \in \overline{\mathsf{HF}} \mid \mathsf{rk}(x) = i\}$ for each $i \in \mathbb{N}$.

At step $n + 1$, the ordered partition $\mathcal{Y}^{n+1}$ is defined as a refinement of $\mathcal{Y}^n$. We say that a block Y_i^n *can be split* if it contains two inequivalent elements with respect to the relation $\sim_\ni$ defined by

$$x \ \sim_\ni \ y \quad \Leftrightarrow \quad \forall j\,(\,Y_j^n \cap x = \emptyset \leftrightarrow Y_j^n \cap y = \emptyset\,). \tag{4}$$

By considering the smallest number h such that Y_h^n can be split and the partition of the block Y_h^n induced by $\sim_\ni$, we proceed by sorting the $\sim_\ni$-equivalence classes of Y_h^n as $Z_0, Z_1, \ldots, Z_m$ ($m \geqslant 1$). Then we put:

$$Y_i^{n+1} \quad = \quad \begin{cases} Y_i^n & \text{if } \ i < h, \\ Z_{i-h} & \text{if } \ h \leqslant i \leqslant h + m, \\ Y_{i-m}^n & \text{if } \ h + m < i. \end{cases} \tag{5}$$

In sight of getting a linear order of $\overline{\mathsf{HF}}$, we define the dyadic relation:

$$x \prec y \quad \Leftrightarrow \quad \exists n\big(\, f(x,n) < f(y,n)\,\big) \tag{6}$$

over $\overline{\mathsf{HF}}$ in terms of the function $f : \overline{\mathsf{HF}} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that $x \in Y^n_{f(x,n)}$.

We are now ready to introduce the extension of the Ackermann function. First, we use the order $\prec$ on $\overline{\mathsf{HF}}$ to assign *positions* to sets in $\overline{\mathsf{HF}}$. We must use *integer* positions, since natural positions are already occupied by well-founded sets.

If $a \in \overline{\mathsf{HF}}$, define:

$$\mathbb{Z}_A(a) \quad = \quad \begin{cases} |\{b \in \mathsf{HF} \mid b \prec a\}| & \text{if } \ a \in \mathsf{HF}, \\ -|\{b \in \overline{\mathsf{HF}} \setminus \mathsf{HF} \mid b \prec a\}| - 1 & \text{if } \ a \in \overline{\mathsf{HF}} \setminus \mathsf{HF}. \end{cases}$$

Let $\mathbb{Q}_2$ be the set of all dyadic numbers; that is, the rational numbers whose binary expansions have finitely many digits. We define a bijection $\mathbb{Q}_A$ from $\overline{\mathsf{HF}}$ to dyadic numbers as follows:

$$\mathbb{Q}_A(a) \quad = \quad \Sigma_{b \in a} 2^{\mathbb{Z}_A(b)}.$$

2. Conclusions

Very much like Ackermann's bijection which it generalizes, the bijection proposed in this work encodes a most basic data type by numbers. This embedding acts as an Occam's razor, by reducing multiplicity to simplicity: in the case of HF-sets, one can implement a full battery of set-handling methods resorting to natural numbers as their internal representation; likewise, one can implement $\overline{\mathsf{HF}}$-sets on top of rational numbers on the ground of the encoding technique proposed above.

It may turn out that other rank notions lead to encodings more satisfactory from a logico-mathematical perspective than the one proposed above, and along the same lines; but the real challenge, we believe, lies in the design of 'light' encoding' techniques that enable one to shift algorithm from one realm (e.g. integers) to another (e.g., special classes of graphs), cf. [Tar09]. When it comes to algorithmic complexity issues, Occam's simplicity does not suffice.

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